CSCI 262
Data Structures

4 – Analysis of Recursive Algorithms,
Binary Search,
Merge Sort

Analysis of

RECURSIVE ALGORITHMS

Recursive Function Analysis

Here's a simple recursive function which raises one number to a (non-negative) power:

```
double power(double n, unsigned k)
    if k == 0 return 1
    return n * power(n, k-1)
```

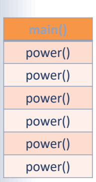
What is the cost of power()?

Analyzing Power

- First, note that we want to analyze power in terms of k, not n (why?)
- Now, ask the following two questions:
  - How much work do we do within power(), excluding the recursive call?
  - How many calls do we make to power()?

Analyzing Power

We can think of this another way by visualizing our call stack, and ask these questions:

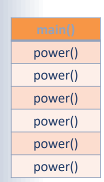


How much work at each level?

How many levels?

Analyzing Power

```
double power(double n, unsigned k)
    if k == 0 return 1
    return n * power(n, k-1)
```



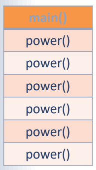
How much work at each level?

One comparison, one multiplication

How many levels?

Analyzing Power

```
double power(double n, int k)
     if k == 0 return 1
     return n * power(n, k-1)
```



How much work at each level?

One comparison, one multiplication

How many levels?

How many times can we subtract 1 before we get to k == 0?

Analyzing Power

Analysis:

2 operations per level * k levels

= 2k operations

In "Big O", we drop constants, so that's O(k).

Analyzing Power 2

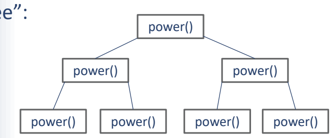
Suppose we try a different approach. This one is doubly-recursive:

```
double power(double n, unsigned k)
    if k == 0 return 1
    else if k == 1 return n
    else return power(n, ⌈k/2⌉) * power(n, ⌊k/2⌋)
```

The expression ⌈x⌉ is called the *ceiling* of x, and means that we round up to the nearest integer. ⌊x⌋ is called the *floor* of x, and means we round down.

Analyzing Power 2

- Now things are more complicated, because each call to power turns into two more calls to power, etc.
- Instead of a stack, we can visualize this as a "call tree":

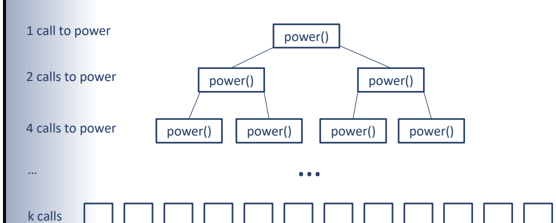


- How many calls to power here?

Analyzing Power 2

- For these kinds of problems, easier to approximate using an ideal case:
  - Assume k is power of 2: $k = 2^p$
  - Now we divide k evenly in half at each level
- How many levels are in our tree?
- How much work is done at each level?

Analyzing Power 2



Analyzing Power 2

We do constant work in power.
So our work is less than or equal to:

some constant * (1 + 2 + 4 + … + k/2 + k)

= some constant * k * (1/k + … 1/4 + 1/2 + 1)

The sum 1 + 1/2 + 1/4 + … 1/k < 2, so our total is

< 2 * some constant * k = O(k), same as before!

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A Smarter Way

Here's a better way:

```
double power(double n, unsigned k)
   if k == 0 return 1
   double m = power(n, ⌊k/2⌋)
   if k is even
        return m * m
   else
        return m * m * n
```

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Correctness

```
double power(double n,
unsigned k)
    if k == 0 return 1
    double m = power(n, ⌊k/2⌋)
       if k is even
          return m * m
       else
          return m * m * n
```

Does this work?

Try it: let k = 11

power (n, 11)

m = power (n, 5)

k is odd so

return (m * m * n) = (n^5 * n^5 * n) = n^{11} ✓

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Analyzing Power 3

Compare to previous version:

- Only 1 recursive call
- Still divide k in half at each step

Now our call “tree” is just a stack again…

But shorter than the first version's stack!

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Analyzing Power 3

How high is the stack?

How many times can you divide a number by 2 before getting to 1?

So the cost of this version is O($\log_2$ k), much better than O(k).

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Searching with

DIVIDE AND CONQUER

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Divide and Conquer

- Split problem into multiple smaller sub-problems
- Solve the sub-problems *recursively*
- Recombine solutions afterwards
- When splitting/recombination can be done efficiently, this approach is a winner

Linear Search

Search for a value in a sorted list.

Obvious approach:

```
// find element k in sorted list x containing n elements
search(x, k)
for i = 1 to n
    if x[i] == k return i
return  NOTFOUND
```

Pseudocode usually starts at index=1, not index = 0

Complexity: O(N)

Binary Search

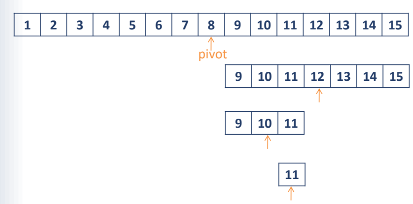
Search for a value in a sorted list.

```
// find element k in sorted list x containing n elements
binary_search(x, k)
   if x is empty
          return NOTFOUND
   pivot = n/2    // look at element halfway through list
   if  x[pivot] == k
          return pivot   // if found, return
   else if k < x[pivot]  // else search left or right sublist
          return binary_search(x[1 : pivot-1], k)
   else
          return binary_search(x[pivot+1 : n], k)
```

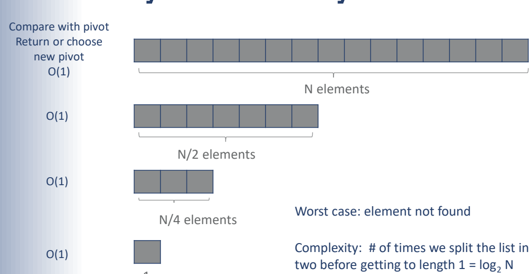

Binary Search Example

Search for a value in a sorted list.

Example: search for 11 in the list 1-15



Analysis of Binary Search



Worst case: element not found

Complexity: # of times we split the list in two before getting to length 1 = $\log_2 N$

Another divide & conquer algorithm:

MERGE SORT

Merge Sort

- Divide and Conquer algorithm for sorting
  - Split input list in half
  - Sort the halves
  - Merge the sorted lists

```
merge_sort(x)
  n = length(x)
  if n == 1 return x
  left = merge_sort(x[1 : n/2])
  right = merge_sort(x[n/2 + 1 : n])
  return merge(left, right)
```

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Merge Sort

- Divide and Conquer algorithm for sorting
  - Split input list in half
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  - Merge the sorted lists

```
merge_sort(x)
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  return merge(left, right)
```

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Merge Sort

```
merge(a, b)
     // treat a, b as stacks or queues
     x = empty list
     loop
          if a is empty
               append b to x, return x
          else if b is empty
               append a to x, return x
          else if top(a) < top(b)
               append pop(a) to x
          else append pop(b) to x
     return x
```

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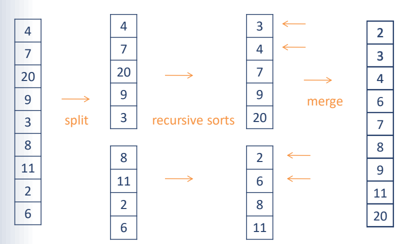
Merge Sort Illustrated



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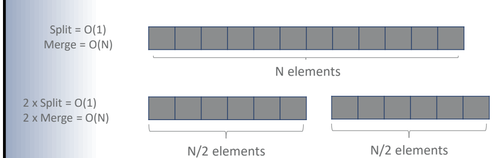
Analysis of Mergesort



Complexity: ?

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Interlude

LOGARITHMS AND BIG O

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About Logarithms

- $\log_b b^k = k$
- For any b, $\log_2 x = \log_b x / \log_b 2$

This shows that the base doesn't matter in "big O" – all bases are just a constant factor from base 2.

- Because "$\log_2 x$" comes up so often, it is often abbreviated to "lg x" in computer science

SORTING IN THE STL

Sorting in Standard Library

- Sorting in the C++ standard library
  - Works on *random access* iterators
  - Works on vectors, strings, and arrays

```
#include <algorithm>
void sort(begin_iterator, end_iterator)
```


sort example

Sorting a vector:

```
#include <algorithm>
…
vector<int> vec = {17, 42, 100, -3, 50};
sort(vec.begin(), vec.end());

for (int n: vec) cout << n << " ";
```

Output:

```
-3 17 42 50 100
```


Another sort example

Sorting a string:

```
#include <algorithm>
…
string s = "Hello, world!";
…
sort(s.begin(), s.end());
cout << s << endl;
```


sort Notes

- Elements of container must be comparable using "<"
  - Depending on application, may be able to overload "<" for items to be sorted
  - Otherwise, have to supply a separate bool valued function as a third parameter to sort:

```
bool rev(int a, int b) {
        return b < a;     // default comparison is a < b
}

int main() {
        vector<int> foo = {16, 4, 23, 1, 2, 17, 6};

        sort(foo.begin(), foo.end());      // {1, 2, 4, 6, 16, 17, 23}
        sort(foo.begin(), foo.end(), rev); // {23, 17, 16, 6, 4, 2, 1}
        return 0;
}
```

Up Next

- Reading: Chapter 12.4 – 12.6, 12.7 optional
- Friday, September 7
  - Lab 3
- Monday, September 10
  - Lab 3 due
  - Project 1 – Image Editor due
  - TBA

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